One-ended subforests and whatnot

Part 6

Local-global bridges.

For pmp actions $\Gamma \overset{\text{pmp}}{\curvearrowright} (X,\mu)$. For any pmp action of a gp Γ on a prob. space (X,μ), for any $f \in L^1(X,\mu)$, e.g. $f := \mathbb{1}_A$, $A \subseteq X$, for any finite $F \subseteq \Gamma$,

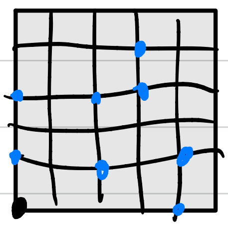
$$\int_X f(x)\, d\mu(x) = \int_X A_f(F,x)\, d\mu(x),$$

where $A_f(F,x) := \frac{1}{|F|} \sum_{\gamma \in F} f(\gamma \cdot x)$.

E.g. $\Gamma := \mathbb{Z}^2$, $F := n \times n$ square, action is free

Then $A_f(F,x)$

$f := \mathbb{1}_A$



Proof. Measure-preserving means

$$\underset{\displaystyle\int \mathbb{1}_A\, d\mu}{\mu(A)} = \underset{\displaystyle\int \mathbb{1}_A(\gamma \cdot x)\, d\mu(x)}{\mu(\gamma^{-1} A)} \qquad \forall \gamma \in \Gamma.$$

$\Rightarrow$ $\forall f \in L^1(X,\mu)$,

$$\int f\, d\mu = \int f(\gamma \cdot x)\, d\mu(x).$$

Then

$$\int f\, d\mu = \frac{1}{|F|} \cdot |F| \cdot \int f\, d\mu = \frac{1}{|F|} \cdot \sum_{\gamma \in F} \int f(\gamma \cdot x)\, d\mu(x)$$

$$= \int A_f(F,x)\, d\mu(x). \qquad \square$$

For pmp finite eq. rel. Let F be a finite pmp eq. rel on (X,μ). Then $\forall f \in L^1(X,\mu)$

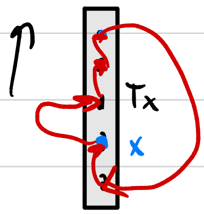
$$\int f\, d\mu = \int A_f[x]_F\, d\mu(x),$$

where $A_f[x]_F := \frac{1}{|[x]_F|} \cdot \sum_{y \in [x]_F} f(y)$.

Proof. $X = \bigsqcup_n X_n$, where $F|_{X_n}$ has exactly n-elements in each class.

We prove this separately for each X_n, so we assume WLOG that each F-class in X has exactly n elements.
Fix a Borel linear order $<$ on X and define:

$$T : X \to X \quad \text{by} \quad x \mapsto \begin{cases} \text{the } <\text{-next pt in } [x]_F & \text{if exists.} \\ \min [x]_F & \text{o.w.} \end{cases}$$



This T defines a free action of $\mathbb{Z}/n\mathbb{Z}$ on X s.t. $E_{\mathbb{Z}/n\mathbb{Z}} = F$. By the local-global bridge for pmp gp actions,

$$\int f \, d\mu = \int \frac{1}{|\mathbb{Z}/n\mathbb{Z}|} \cdot \sum_{\gamma \in \mathbb{Z}/n\mathbb{Z}} f(\gamma \cdot x) \, d\mu(x) = \int A_f[x]_F \, d\mu(x).$$

□

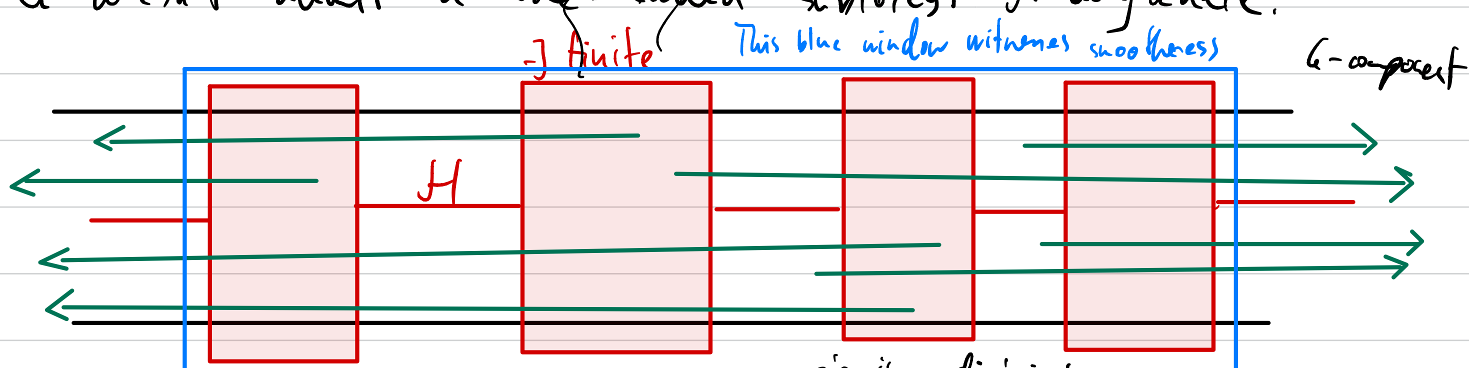
Prop. Let G be a loc. finite Borel graph on a stand. Borel space X. Suppose G is 2-ended. Then

(a) G is Borel hyperfinite, i.e. $G = \bigcup\!\!\uparrow_n G_n$, where G_n is a Borel component-finite graph.

(b) If G admits a one-ended spanning Borel subforest, then G is smooth ($:=$ $\exists$ Borel choice of one point from each G-component).

In particular, if μ is an E_G-invariant prob. meas. on X, then G doesn't admit a one-ended subforest μ-anywhere.

Proof.



Let $\mathcal{S}$ be a Borel collection of pairwise disjoint finite G-connected subsets of X

that is maximal (exists using Feldman–M gone).
We may define the neighbourhood graph $\mathcal{H}$ on $\mathcal{S}$ by

$$S_1 \mathcal{H} S_2 \;:\Longleftrightarrow\; S_2 \text{ is in an infinite component of } G - S_1 \text{ and } \nexists \text{ no other } S' \in \mathcal{S} \text{ in between.}$$

Modulo smooth, $\mathcal{H}$ is 2-regular.

(a) Using, say, the marker lemma, we get $\mathcal{S}_n \subseteq \mathcal{S}$ s.t. $\bigcap\limits_n^{\downarrow} \mathcal{S}_n = \emptyset$.

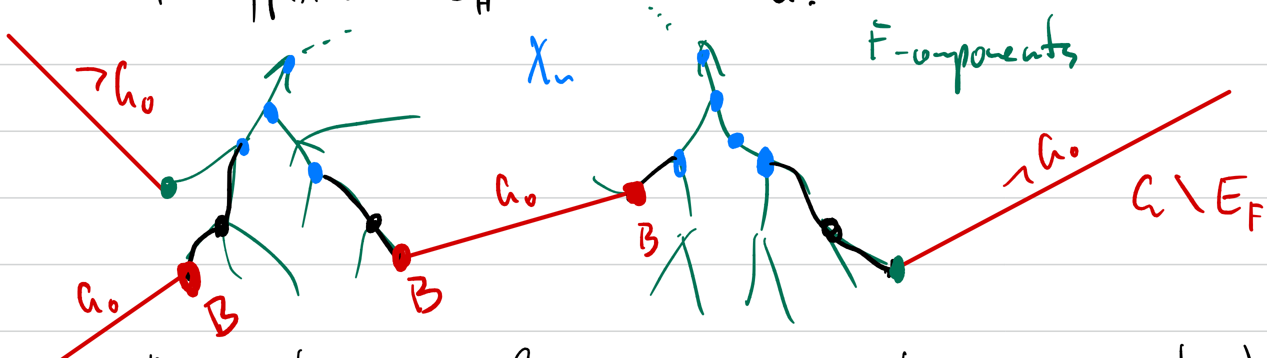
Then $G_n := G - \bigcup \mathcal{S}_n$ is component finite and $G = \bigcup\limits_n^{\uparrow} G_n$.

(b) Proof by picture. $\square$

<u>1% lemma.</u> Let G be a loc. finite pmp graph on (X, μ). G is μ-nowhere hyperfinite $\Longleftrightarrow$ $\forall \varepsilon > 0$ (1%) $\exists$ $A \subseteq X$ (G-complete section) with $\mu(A) < \varepsilon$ s.t. $G|_A$ is μ-nowhere hyperfinite.

Proof. $\Leftarrow$. G μ-hyperfinite $\Rightarrow$ $G|_A$ is μ-hyperfinite.

$\Rightarrow$. By the prev. prop, G is nowhere 2-ended, hence admits a one-ended Borel subforest F. Moreover, by measure-exhaustion, we may take F s.t. E_F is maximal among all hyperfinite E_H for $H \subseteq G$.



By shaving leaves n-many times, we get X_n, so $\bigcap\limits_n X_n = \emptyset$. $\exists$ n s.t. $\mu(X_n) < \frac{1}{2}\varepsilon$.
Note that $E_F|_{X_n} = E_{F|_{X_n}}$, i.e. each E_F class is

X_n is still F-connected. Because G is μ-nowhere hyperfinite and E_F is μ-hyperfinite, a.e. G-component must contain ≥ 2 F-components. G_0 meets every F-component
By marker lemma, $\exists$ $G_0 \subseteq G \setminus E_F$ s.t. the set $B \subseteq X$ of all points incident to G_0 is small: $\mu(B) < \frac{1}{n}\frac{1}{2}\varepsilon$.

For each $x \in B$, let P_x be the unique F-path from x to X_n. Then $|P_x| \leq n$, so $\overline{B} := \bigcup_{x \in B} P_x$, so $\mu(\overline{B}) \leq n \cdot \mu(B) < \frac{1}{2}\varepsilon$. Let $A := \overline{B} \cup X_n$.

Because $G|_A$-connected components contain the G_0 edges, each $G|_A$-component contains ≥ 2 $E_F|_A$-classes.
By the maximality of E_F, $G|_A$ is μ-nowhere hyperfinite. $\square$